ISRAEL JOURNAL OF MATHEMATICS 99 (1997), 55-67

ENTROPY RIGIDITY FOR SEMISIMPLE GROUP ACTIONS

BY

NANTIAN QIAN

Department of Mathematics, Yale University
New Haven, CT 06520, USA
e-mail: qian@math.yale.edu

AND

ROBERT J. ZIMMER

Department of Mathematics, University of Chicago
Chicago, IL 60637, USA
e-mail: r_zimmer@math.uchicago.edu

ABSTRACT

We show for the standard actions of higher rank semisimple Lie groups and their discrete subgroups on compact manifolds that the entropy is invariant under small perturbations.

1. Introduction and the statement of results

Let G be a semisimple Lie group with finite center and all simple factors of real rank at least 2, and $\Gamma \subset G$ be a lattice in G. In [Z4] three types of algebraic smooth volume preserving actions of Γ on a compact smooth manifold M have been described: (1) Isometric actions; (2) Left translations on compact quotients H/Λ via homomorphism $\Gamma \to H$, where H is a connected Lie group, $\Lambda \subset H$ is a co-compact lattice; (3) Affine actions on compact nilmanifolds. For simplicity we call them type 1, type 2 and type 3 actions in this paper, respectively. New examples of smooth volume preserving Γ actions can be obtained by simple algebraic constructions such as products, finite extensions, and compact quotients from this list. We note that G also acts on H/Λ by left translations via homomorphism $G \to H$ (we also say that this action is of type 2). We later refer to all actions described here as the **standard** higher rank group actions.

Received October 6, 1995

As is well-known, for an arbitrary measure preserving diffeomorphism on a compact manifold, the Lyapunov exponents (functions defined almost everywhere) are not continuous functions, and do not change continuously under small perturbation of the diffeomorphism. One anticipates rigidity results for higher rank group actions. For actions ρ_0 (necessarily of Γ) preserving a smooth riemannian metric (and hence of trivial Lyapunov exponents and zero entropy), the second author showed in [Z2], among other results, that for any measure preserving action ρ C^1-close to ρ_0, the Lyapunov exponents as well as entropy are unchanged. We show in this paper that for all standard higher rank group actions, the Lyapunov exponents (which are constants) as well as the entropy for the individual element are locally rigid (i.e., they are unchanged under small perturbations of the actions).

The local rigidity of the Lyapunov exponents and the entropy holds true for a larger class of higher rank group actions called actions with **point Mather spectrum** (Definition 2.4). The standard higher rank group actions, tangentially flat actions and almost tangentially flat actions (Definition 2.4) are actions with point Mather spectrum, and they preserve absolutely continuous measures equivalent to some Lebesgue measure.

THEOREM A: *Let G and Γ be as in* (A1), (A3) *of* §3 *and ρ_0 a C^1-action of G (or Γ) on a smooth compact manifold M with point Mather spectrum, preserving Lebesgue measure. Let ρ be a C^1 action sufficiently C^1-close to ρ_0* and m *a ρ-invariant probability measure. Then the set of Lyapunov exponents of $\rho(g)$ is the same as that of $\rho_0(g)$ for all $g \in G$ (or $g \in \Gamma$) (Theorems* 3.2, 3.3*).*

One corollary of Theorem A is the rigidity of the measure-theoretic entropy for the individual elements in the group.

THEOREM B: *Let G, Γ and m be as in Theorem* A, *and let ρ be a $C^{1+\epsilon}$ action sufficiently C^1-close to a $C^{1+\epsilon}$ volume preserving action ρ_0 ($\epsilon > 0$) with point Mather spectrum. If in addition m is absolutely continuous with respect to Lebesgue measure, then the measure-theoretic entropy $h_m(\rho(g)) = h(\rho_0(g))$ for all $g \in G$ (or $g \in \Gamma$). Here $h(\rho_0(g))$ is both the topological entropy and the measure-theoretic entropy (with respect to the invariant volume form) for $\rho_0(g)$ (Theorem* 3.4 (2)*).*

Theorem B strengthens a result in [Z3] that such entropy for individual elements has only finite many possibilities (9.4.13, 9.4.16 of [Z3]). This also partially answers a question raised in a lecture by Furstenberg [Fu] concerning the continuity of measure-theoretic entropies for individual elements in higher rank group actions preserving Lebesgue measures. We point out that it is not known whether the perturbations of the standard actions have point Mather spectrum.

ACKNOWLEDGEMENT: The first author is grateful to the University of Chicago and the University of North Carolina at Chapel Hill for their hospitality and financial support, and would like to thank P. Eberlein for helpful discussions.

2. Lyapunov exponents and Mather spectrum

Let M be a compact manifold and $f\colon M \to M$ be a C^1 diffeomorphism. For every $x \in M$ and $v \in TM_x - \{0\}$, we define $\chi^+(x,v,f) = \limsup_{n\to\infty} \frac{1}{n} \ln \|Df^n(x)v\|$ and $\chi^-(x,v,f) = \limsup_{n\to-\infty} \frac{1}{n} \ln \|Df^n(x)v\|$.

LEMMA 2.1: *Suppose that either*

(1) *f is an analytic diffeomorphism such that $f^n \neq \mathrm{id}$, or*
(2) *there exists a dense set $D \subset M$ with the following property: for all $x \in D$, there exists $v \in TM_x - \{0\}$ such that $\chi^+(x,v,f) \neq 0$ or $\chi^-(x,v,f) \neq 0$.*

Then the set of nonperiodic points is dense in M.

Proof: Let P_n denote the set of points $x \in M$ such that $f^n(x) = x$. Since P_n is a closed set, by Baire Category Theorem it is enough to show that P_n has no interior points (since $\bigcup_n P_n$ is of first category). Suppose the contrary: let U be a nonempty open set such that $U \subset P_n$, and hence $f^n|_U = \mathrm{id}|_U$. In case (1), this implies that $f^n = \mathrm{id}$, contrary to the assumption of f in (1). In case (2), since D is dense in M, there exists $p_0 \in P_n \cap D$. Since $f^n|_U = \mathrm{Id}_U$, we have $Df^n_{p_0} = \mathrm{Id}\colon T_{p_0} \to T_{p_0}$. This implies that $\chi^+(p_0,v,f) = 0$ and $\chi^-(p_0,v,f) = 0$ for $p_0 \in D$, all $v \in TM_{p_0} - \{0\}$, contrary to the assumption in (2). ∎

We remark that all standard actions ρ_0 are analytic. Therefore for any $g \in G$ or Γ satisfying $\rho_0(g^n) \neq \mathrm{Id}$ for any $n \in \mathbb{Z}^+$, non-periodic points of $\rho_0(g)$ are dense in M.

For a C^1 diffeomorphism $f\colon M \to M$ on a compact smooth manifold with riemannian metric d given by inner product $< \cdot, \cdot >$, we may define an operator f_* on the space $\mathrm{Vec}^0(M)$ of C^0 vector fields by the formula $f_*v(x) = (Df)(v(f^{-1}(x)))$. According to Mather ([Mat], see also [Pel]), the operator L obtained by complexification of f_* possesses a spectrum consisting of all the points between full circles, provided that the non-periodic points of f are dense in M. Each of the connected components of the spectrum is thus a $[\lambda_i, \mu_i]$-ring (a closed annulus about the origin with radii λ_i and μ_i, $\lambda_i \leq \mu_i$), and corresponding to each $[\lambda_i, \mu_i]$-ring there is a continuous f-invariant subbundle E_i in TM such that for every $\delta > 0$, there exists $q_i, q_i' > 0$, such that $q_i(\lambda_i - \delta)^n \|v_i\| \leq \|Df^n v_i\| \leq q_i'(\mu_i + \delta)^n \|v_i\|$ for all $v_i \in E_i(x), x \in M, n > 0$ ([Pel], [BP]). Therefore, it is easy to see that $\ln \lambda_i \leq \chi^{\pm}(x, v_i, f) \leq \ln \mu_i$.

We summarize the facts in the following theorem.

THEOREM 2.2 ([Pel]): *Let $f \in \mathrm{Diff}^1(M)$ and assume that the non-periodic points of f are dense in M. Then there exist a set of rings $R_1, \dots, R_k$ (where each R_i is the $[\lambda_i, \mu_i]$-ring), corresponding to each ring R_i an f-invariant tangent subbundle E_i, such that*

1. *E_i is a continuous vector bundle;*
2. *For every $\delta > 0$, there exists $q_i, q_i' > 0$, $q_i(\lambda_i - \delta)^n \|v_i\| \leq \|Df^n v_i\| \leq q_i'(\mu_i + \delta)^n \|v_i\|$ and $q_i(\mu_i + \delta)^{-n} \|v_i\| \leq \|Df^{-n} v_i\| \leq q_i'(\lambda_i - \delta)^{-n} \|v_i\|$ for all $v_i \in E_i(x), x \in M, n > 0$;*
3. *$\ln \lambda_i \leq \chi^{\pm}(x, v_i, f) \leq \ln \mu_i$.* ∎

Let f have dense non-periodic points and $k, \lambda_i(f) = \lambda_i, \mu_i(f) = \mu_i, E_i(f) = E_i$ be as in Theorem 2.2; let $I_i(f) = [\lambda_i(f), \mu_i(f)]$, $i = 1, \dots, k$. We define the **Mather spectrum** of f to be the set of pairs $\{(I_1(f), E_1(f)), \dots, (I_k(f), E_k(f))\}$ and denote it by $\mathrm{Spect}_m(f)$. The following result of Pesin asserts that the Mather spectrum has a certain continuity property.

THEOREM 2.3 ([Pel]): *Let $\mathrm{Spect}_m(f)$ consist of at least two pairs. Then for every $\epsilon > 0$ there exists a neighborhood η of f in $\mathrm{Diff}^1(M)$ such that, for any $g \in \eta$,*

$$\begin{aligned}\mathrm{Spect}_m(g) = \{&(I_{1,1}(g), E_{1,1}(g)), \dots, (I_{1,i_1}(g), E_{1,i_1}(g)),\\ &\dots,\\ &(I_{k,1}(g), E_{k,1}(g)), \dots, (I_{k,i_k}(g), E_{k,i_k}(g))\},\end{aligned}$$

where for each j and $l = 1, \dots, i_j$, $I_{j,l}(g) = [\lambda_{j,l}(g), \mu_{j,l}(g)]$ and $\lambda_{j,1} \leq \mu_{j,1} < \lambda_{j,2} \leq \mu_{j,2} < \cdots < \lambda_{j,i_j} \leq \mu_{j,i_j}$, with the following properties:

1. *$|\lambda_j(f) - \lambda_{j,1}(g)| \leq \epsilon$, $|\mu_j(f) - \mu_{j,i_j}(g)| \leq \epsilon$ for $j = 1, \dots, k$;*

(2) *Let d be the metric measuring the distance between subbundles induced by riemannian metric. Then* $d(E^j(f), \bigoplus_{s=1}^{i_j} E_{j,s}(g)) \leq \epsilon$. ∎

We remark that for C^1 diffeomorphisms f whose non-periodic points are not necessarily dense, we also define the Mather spectrum of f provided that f preserves a continuous riemannian metric or has subexponential growth with respect to a riemannian metric for the tangent map (i.e., for every $\delta > 0$, there exists $q, q' > 0$, $q(1-\delta)^n\|v\| \leq \|Df^n v\| \leq q'(1+\delta)^n\|v\|$ and $q_i(1+\delta)^{-n}\|v\| \leq \|Df^{-n}v\| \leq q'(1-\delta)^{-n}\|v\|$ for all $v \in TM_x, x \in M, n > 0$), by simply letting $\mathrm{Spect}_m(f) = ([1,1], TM)$. It is clear that for such f, $\chi^{\pm}(x, v, f) = 0$ for all $x \in M$ and $v \in TM_x$, and for any diffeomorphism h C^1-close to f, $\chi^{\pm}(x, v, h)$ is close to 0 uniformly in $x \in$ and $v \in TM_x$.

Definition 2.4: (a) Let $f \in \mathrm{Diff}^1(M)$ either have dense non-periodic points in M or have subexponential growth with respect to a riemannian metric for the tangent map.

(1) f has **point Mather spectrum** if $\lambda_i = \mu_i$ for all $i = 1, \ldots, k$.
(2) f is **tangentially flat** if there exists a continuous framing σ (i.e., a set of n-continuous non-vanishing vector fields $\{X_1, \ldots, X_n\}$ on M) such that $Df \circ \sigma(x) = \sigma(f(x))A$ for a constant matrix A.

(b) Let ρ be a C^1-action ρ of G (or Γ) on M.

(1) ρ has **point Mather spectrum** if, for each $g \in G$ (or $g \in \Gamma$), $\rho(g)$ has point Mather spectrum; in particular, $\rho(g)$ either has dense non-periodic points or has subexponential growth with respect to some riemannian metric.
(2) ρ is **tangentially flat** associated with a homomorphism $\pi: G \to \mathrm{GL}(n, \mathbb{R})$ if there exists a continuous framing σ such that $D\rho(g) \circ \sigma(x) = \sigma(\rho(g)(x))\pi(g)$ for all $g \in G$.
(3) ρ is **almost tangentially flat** if there is a compact Lie group C, and a principal C-bundle E over M on which G (or Γ) acts by principal bundle automorphisms such that the induced action on M is the original action ρ, and the action on E is tangentially flat.

We remark that f has point Mather spectrum if and only if there exist finitely many positive numbers $\lambda_1, \ldots, \lambda_k$ ($k \geq 1$) such that, for each $i = 1, \ldots, k$, there is an f-invariant continuous tangent subbundle E_i, such that for every $\delta > 0$, there exists $q_i, q_i' > 0$, such that $q_i(\lambda_i - \delta)^n \|v_i\| \le \|Df^n v_i\| \le q_i'(\lambda_i + \delta)^n \|v_i\|$ and $q_i(\lambda_i + \delta)^{-n} \|v_i\| \le \|Df^{-n} v_i\| \le q_i'(\lambda_i - \delta)^{-n} \|v_i\|$ for all $v_i \in E_i(x), x \in M, n > 0$.

Therefore if a diffeomorphism f (or an action ρ) is tangentially flat then it has point Mather spectrum. It is also clear that if an action is tangentially flat then it is almost tangentially flat. Moreover, we have the following.

LEMMA 2.5: *Let $f \in \mathrm{Diff}^1(M)$ either have dense non-periodic points in M or have subexponential growth with respect to a riemannian metric for the tangent map.*

(1) *If f has point spectrum, then $\chi^{\pm}(x, v_i, f) = \ln \lambda_i$ and $\prod \lambda_i^{\dim(E_i)} = 1$ for all $x \in M, v_i \in E_i$ provided that for each $i = 1, \ldots, k$, E_i is orientable;*
(2) *If f is tangentially flat, then the matrix A associated to f with respect to the linearizing framing is in $\mathrm{SL}^{\pm}(n, \mathbb{R})$ and hence f preserves a continuous volume form.*

Proof: Let f have point spectrum and $\mathrm{Spect}_m(f) = \{(\lambda_1, E_1), \ldots, (\lambda_k, E_k)\}$ (we set $\lambda_i = [\lambda_i, \lambda_i]$). It is clear that $\chi^{\pm}(x, v_i, f) = \ln \lambda_i$. We also assume that for every $\delta > 0$, there exists $q_i, q_i' > 0$, $q_i(\lambda_i - \delta)^n \|v_i\| \le \|Df^n v_i\| \le q_i'(\lambda + \delta)^n \|v_i\|$ for all $v_i \in E_i(x), x \in M, n > 0$. We choose a continuous metric on M such that E_i are mutually orthogonal, an orthonormal basis on each of E_i and use the dual of the basis to construct a top continuous form α_i on E_i. Define a continuous volume form $\alpha = \bigwedge_1^k \alpha_i$. Now, using the formula of integration for the change of variable $y = f(x)$ we obtain

$$\int_M d\alpha(x) = \int_{f^n M} d\alpha(y) = \int_M |\det(Df^n)(x)| d\alpha(x).$$

Since $\prod(\lambda_i - \delta)^{\dim(E_i)} \le |\det(Df^n)(x)| \le \prod(\lambda_i + \delta)^{\dim(E_i)}$ we conclude that if $\prod \lambda_i^{\dim(E_i)} \neq 1$ we may change the volume of a compact manifold M by a diffeomorphism, which is impossible.

The second assertion is a direct corollary of the first, since $\det(A)$ is the determinant of the tangent map Df. ∎

Similar statements can also be made for almost tangentially flat actions, the proof of which is left to the reader.

LEMMA 2.6: *Let ρ be an almost tangentially flat action of G or Γ on M. Then ρ has point Mather spectrum, and preserves an absolutely continuous measure equivalent to some Lebesgue measure.*

Examples 2.7: (1) The standard higher rank group actions of type 1–3 are actions with point Mather spectrum. (2) The standard higher rank group actions of type 2 and type 3 are tangentially flat actions. (3) Let H be a connected Lie group, D be a cocompact lattice of H, and C be a compact subgroup of H. Let the action of G or Γ on $C\backslash H/D$ be given by a homomorphism into H whose image is centralized by C. This action is not tangentially flat. However, it is almost tangentially flat.

It is easy to see that the finite products, finite extensions and finite quotients obtained from known examples of the actions with point Mather spectrum preserving absolutely continuous measure also have such properties. Since all actions in Examples 2.7 preserve some absolutely continuous measure, we conclude that all standard higher rank group actions have point Mather spectrum, and preserve some absolutely continuous measure.

3. Local rigidity of the Lyapunov exponents of actions with point Mather spectrum

In the rest of the paper, unless otherwise specified, we assume that:

(A1) G is a finite product $\prod G_i$ where each G_i is the group of k_i-points of a k_i-simple connected k_i-group $\mathbf{G}_i$ of k_i-rank at least 2, where k_i is a local field of characteristic 0; for $k_i = \mathbb{R}$, we allow G_i to be any connected semisimple Lie group with finite center and all simple factors of $\mathbb{R}$-rank at least 2;

(A2) $\tilde{\mathbf{G}}_i$ is the algebraic simply connected covering of $\mathbf{G}_i$ in the algebraic group case, $\tilde{G}_i$ is the k_i-points of it; if $k_i = \mathbb{R}$ and G_i is only a connected Lie group, we let $\tilde{G}_i$ be the universal covering Lie group of G_i, $\tilde{\mathbf{G}}_i$ the maximal algebraic factor of $\tilde{G}_i$ (since G_i modulo the center is the (Hausdoff) connected component of the real points of an algebraic $\mathbb{R}$-group $\mathbf{G}_i'$, the algebraic universal cover of $\mathbf{G}_i'$ is $\tilde{\mathbf{G}}_i$); $\tilde{\mathbf{G}}$ is the finite product $\prod \tilde{\mathbf{G}}_i$, $\tilde{G}$ is the finite product $\prod \tilde{G}_i$;

(A3) Γ is a lattice in G;

(A4) $\mathbf{H}$ is an $\mathbb{R}$-algebraic group, $\mathbf{H}^0_{\mathbb{R}}$ is the real points of the Zariski connected component $\mathbf{H}^0$ of $\mathbf{H}$;

(A5) M is an n-dimensional compact smooth manifold, m is a probability measure on M;

(A6) π_0 is a continuous homomorphism of G or Γ to $\mathrm{GL}(n, \mathbb{R})$, $\tilde{\pi}_0$ is the lift of π_0 to $\tilde{G}$;

(A7) ρ_0 is C^1-action of G or Γ (depending on the context) on M with point

Mather spectrum.

We now choose a framing on M in the following way. If there exists a tangentially flat action on M, we let the framing be the continuous framing which gives the tangentially flat structure. Otherwise, we let the framing be a measurable framing such that the derivative cocycle for a C^1-action is tempered with respect to this framing (i.e., for any $g \in G$ (or Γ) and $n \in \mathbb{Z}$, $\|\alpha(\cdot, \rho_1(g^n))\| \in L^\infty(M, m)$). If no mention is made of a framing in a statement, we always assume that the framing is the framing we choose above.

In this section, we prove that the Lyapunov exponents of the C^1-actions with point Mather spectrum of G or Γ on smooth compact manifold M are locally rigid. The proof of this fact relies on continuity properties of Mather spectrum (Theorem 2.3) and the following corollary of cocycle superrigidity. We state it for the convenience of the reader.

THEOREM 3.1: *Let G and Γ be as in* (A1) *and* (A3). *Let ρ be a C^1 ergodic action of G (or Γ) on (M, m) preserving a probability measure m and α be the derivative cocycle. Then there exists an associated quadruple $(\mathbf{H}, \beta, \tilde{\pi}, b)$ such that:*

1. *$\mathbf{H}$ is an algebraic $\mathbb{R}$-group and the algebraic hull of α is $\mathbf{H}_{\mathbb{R}}$;*
2. *β is a cocycle over ρ taking values in $\mathbf{H}_{\mathbb{R}}$ and α is equivalent to β;*
3. *for the action ρ' on $M \times \mathbf{H}_{\mathbb{R}}/\mathbf{H}^0_{\mathbb{R}}$ defined by $(x, [h], g) \mapsto (\rho(g)x, \beta(x, g)[h])$, the algebraic hull of β' (a cocycle over ρ' defined by $((x, [h]), g) \mapsto \beta(x, g))$ is $\mathbf{H}^0_{\mathbb{R}}$;*
4. *let $\tilde{G}$ be as in* (A2). *Then β' is equivalent to a tempered cocycle defined by δ: $(x, [h], g) \mapsto \tilde{\pi}(\tilde{g})b(x, [h], \tilde{g})$, where $\tilde{\pi}$ is a homomorphism from $\tilde{G}$ to $\mathbf{H}^0_{\mathbb{R}}$ (which factors to a rational homomorphism of the maximal algebraic factor of $\tilde{G}$) and $b(x, [h], \tilde{g})$ is a measurable map taking values in a compact normal subgroup of $\mathbf{H}^0_{\mathbb{R}}$ ($\tilde{g}$ projects to g under the covering map);*
5. *α' (a cocycle over ρ' defined by $(x, [h], g) \mapsto \alpha(x, g))$ is equivalent to δ.*

We make a comment about this theorem. Let $\mathbf{H}$ be an algebraic $\mathbb{R}$-group such that $\mathbf{H}_{\mathbb{R}}$ is the algebraic hull of α. Then $\mathbf{H}$ is reductive and the real points of the center $(Z(\mathbf{H}))_{\mathbb{R}}$ of $\mathbf{H}$ is compact (by Zimmer [Z5] for G and cocompact Γ, and Lewis [Le] for non-cocompact Γ). In the case that G is the real point of a connected almost $\mathbb{R}$-simple $\mathbb{R}$-group of $\mathbb{R}$-rank≥ 2, the proof can be found in [Z3] (Theorem 9.4.12, together with the result about the algebraic hull of the derivative cocycle mentioned above). In the case that G is a connected semisimple Lie group with finite center such that each of the simple components of which has $\mathbb{R}$-rank at least 2, the theorem is a corollary of Theorem 2.2 of [Z1] (without the assumption of the irreducible ergodicity of the action). In the most general case (G and Γ as in (A1) and (A3)), we also have the cocycle superrigidity result for ergodic actions of G as in the Theorem 2.2 of [Z1], and the theorem is its corollary.

Let f be a C^1-diffeomorphism preserving a probability measure m. The Oseledec multiplicative ergodic theorem [O] asserts that there exists a measurable decomposition of the tangent bundle $TM = E_1 \oplus \cdots \oplus E_s$, such that (1) for m-almost all $x \in M$, all $v \in TM_x - \{0\}$, $\lim_{n\to\pm\infty} \frac{1}{n}\ln\|Df^n(v)\|$ exist (hence they are equal to $\chi^{\pm}(x,v,f)$); (2) for m-almost all $x \in M$, all $v \in TM_x - \{0\}$, $\chi^+(x,v,f) \geq \chi^-(x,v,f)$ and equality holds iff $v \in E_i$ for some i. Such $x \in M$ is said to be a regular point. We remark that if f is tangentially flat, then $E_1,\ldots,E_s$ correspond to different absolute values of eigenvalues. For a regular point $x \in M$, we denote by $X(x,f)$ the multiset of the Lyapunov exponents of f at x including the multiplicity. I.e.,

$$X(x,f) = \{\underbrace{\chi(x,v_1,f),\ldots,\chi(x,v_1,f)}_{\dim(E_1)},\ldots,\underbrace{\chi(x,v_k,f),\ldots,\chi(x,v_k,f)}_{\dim(E_k)}\},$$

where $v_i \in E_i - \{0\}$. We also denote by $X(A)$ the multiset $\{\ln(|\kappa_1|),\ldots,\ln(|\kappa_n|)\}$, where κ_i are all the eigenvalues of matrix A including the multiplicity. Two such multisets are said to be equal if every number in one multiset appears in the other with equal multiplicities.

THEOREM 3.2: *Let G be as in* (A1) *and ρ_0 be a C^1-action with point Mather spectrum, preserving a measure m_0 equivalent to Lebesgue measure.*

(1) *For any ρ_0-invariant, ergodic components m_0^1, m_0^2 of m_0, let the quadruples $(\mathbf{H}_{m_0^i}, \beta_{m_0^i}, \tilde{\pi}_{m_0^i}, b_{m_0^i})$ be associated with ρ_0, m_0^i (i=1,2) as in Theorem 3.1. Then $X(\tilde{\pi}_{m_0^1}(g)) = X(\tilde{\pi}_{m_0^2}(g))$ (denoted by $X_0(g)$). Moreover, $X(x,\rho_0(g)) = X_0(g)$ for all $g \in G$, m_0-almost every $x \in M$.*
(2) *There exists a neighborhood η of ρ_0 in $R(G,\mathrm{Diff}^1(M))$ such that for any $\rho \in \eta$, any ρ-invariant probability measure m, any $g \in G$ and m-almost every $x \in M$, $X(x,\rho(g)) = X_0(g)$.*

We recall for any topological group K, $R(K,\mathrm{Diff}^1(M))$ is the space of all continuous homomorphisms from K to $\mathrm{Diff}^1(M)$ with the open-compact topology (i.e., $\rho_n \to \rho'$ iff $\rho_n(g) \to \rho'(g)$ in C^1-topology for all $g \in K$ and the convergence is uniform on compact sets).

Proof: For two real representations $\tilde{\pi}_0$ and $\tilde{\pi}_1$ of $\tilde{G}$ on $\mathbb{R}^n$ ($n = \dim(M)$), we have either (i) $X(\tilde{\pi}_0(\tilde{g})) = X(\tilde{\pi}_1(\tilde{g}))$ for all $\tilde{g} \in \tilde{G}$ or (ii) there exists $\tilde{g} = \tilde{g}(\pi_0, \pi_1) \in \tilde{G}$ such that $X(\tilde{\pi}_0(\tilde{g})) \neq X(\tilde{\pi}_1(\tilde{g}))$.

We say that $\tilde{\pi}_0$ is equivalent to $\tilde{\pi}_1$ if (i) happens, and denote the equivalence class containing $\tilde{\pi}_0$ by $[\tilde{\pi}_0]$. Let $\Phi = \{[\tilde{\pi}_0], [\tilde{\pi}_1], \ldots, [\tilde{\pi}_s]\}$ be the (finite) set of all equivalence classes of representations of $\tilde{G}$ on $\mathbb{R}^n$. Then for each $i = 1, \ldots, s$ we have $g_i \in G$ such that $X(\tilde{\pi}_0(\tilde{g}_i)) \neq X(\tilde{\pi}_i(\tilde{g}_i))$.

(1) For any C^1-action $\bar{\rho}$ of G on M preserving a probability measure $\bar{m}$, $g \in G$ and a $\mathrm{GL}(n, \mathbb{R})$-valued cocycle α over $\rho(g)$, we define $e_{\bar{\rho}}(\alpha, g)(x)$ as $\lim_{n\to\infty} \frac{1}{n} \ln^+(\|\alpha(x, \bar{\rho}(g^n))\|)$ if it exists. If α is the derivative cocycle, then α is tempered and hence $e(\alpha, \bar{\rho}(g))(x)$ exists for $\bar{m}$-almost every $x \in M$ and hence for almost every regular point. Without loss of generality, we assume for every regular point the above limit exists. It is well-known that the sum of the first p largest Lyapunov exponents of derivative cocycle α at regular points x over $\bar{\rho}(g)$ is $e_{\bar{\rho}}(\wedge^p \alpha)(x)$ (see (3.2) of [Ru1]).

Let α', ρ_0' and the quadruples $(\mathbf{H}_{m_0^1}, \beta_{m_0^1}, \tilde{\pi}_{m_0^1}, b_{m_0^1})$ associated with ρ_0, m_0^1 be as in Theorem 3.1. It is obvious that for m_0^1-almost every $x \in M$ and all $[h] \in (\mathbf{H}_{m_0^1})_{\mathbb{R}}/(\mathbf{H}_{m_0^1})^0_{\mathbb{R}}$, $e_{\rho_0}(\wedge^p\alpha)(x) = e_{\rho_0'}(\wedge^p\alpha')(x, [h])$. Since $\wedge^p\alpha'$ and $\wedge^p\delta$ are equivalent (Theorem 3.1 (5)) tempered cocycles, $e_{\rho_0'}(\wedge^p\alpha')(x, [h]) = e_{\rho_0'}(\wedge^p\delta)(x, [h])$ where δ is as in Theorem 3.1 (4) (see, for example, 9.4.7 of [Z3]). Since for m_0^1-almost all $(x, [h]) \in M \times \mathbf{H}_{\mathbb{R}}/\mathbf{H}^0_{\mathbb{R}}$, $e_{\rho_0'}(\wedge^p\delta)(x, [h]) = \max\{\ln(|\lambda|)\colon \lambda$ is an eigenvalue for $\wedge^p\tilde{\pi}(\tilde{g})\}$, we obtain that $e(\wedge^p\alpha')(x, [h]) = \max\{\ln(|\lambda|)\colon \lambda$ is an eigenvalue for $\wedge^p\tilde{\pi}(\tilde{g})\}$. It follows that $X(x, \rho_0(g_i)) = X(\tilde{\pi}_{m_0^1}(\tilde{g}_i))$ for m_0^1-almost every $x \in M$. The same argument implies that $X(x, \rho_0(g_i)) = X(\tilde{\pi}_{m_0^2}(\tilde{g}_i))$ for m_0^2-almost every $x \in M$.

It follows directly from the definition of point Mather spectrum that for each $g \in G$, the multiset of Lyapunov exponents $X(x, \rho_0(g))$ for $\rho_0(g)$ exists everywhere and is independent of x. In particular, $\tilde{\pi}_{m_0^1}$ and $\tilde{\pi}_{m_0^2}$ are equivalent. Without loss of generality, we assume that the equivalence class is $[\tilde{\pi}_0]$ for a homomorphism $\tilde{\pi}_0\colon \tilde{G} \to \mathrm{GL}(n, \mathbb{R})$.

(2) Without loss of generality we assume that m is ergodic. By Theorem 2.3, there exists a neighborhood η of ρ_0 such that for all $\rho \in \eta$, and any regular point $x \in M$, $X(x, \rho(g_i)) \neq X(\tilde{\pi}_i(\tilde{g}_i))$ for $i \geq 1$. Let the quadruple

$(\mathbf{H}, \beta, \tilde{\pi}, b)$ be associated with ρ. It follows from the argument as in (1) that $X(x, \rho(g_i)) = X(\tilde{\pi}(\tilde{g}_i))$ for the representation $\tilde{\pi}$, and therefore, $\tilde{\pi}$ and $\tilde{\pi}_0$ are in the same equivalence class $[\tilde{\pi}_0]$. It is then clear that for any $\rho \in \eta$, any ρ-invariant Borel probability measure m, any $g \in G$ and m-almost every $x \in M$, $X(x, \rho(g)) = X(\tilde{\pi}_0(\tilde{g})) = X_0(g)$. ∎

For higher rank lattices, we have a similar result.

THEOREM 3.3: *Let Γ be as in (A3) and ρ_0 be a C^1-action with point Mather spectrum, preserving a measure m_0 equivalent to Lebesgue measure.*

(1) *For any ρ_0-invariant, ergodic component m_0^1, m_0^2 of m_0, let the quadruples $(\mathbf{H}_{m_0^i}, \beta_{m_0^i}, \tilde{\pi}_{m_0^i}, b_{m_0^i})$ be associated with ρ_0, m_0^i (i=1,2) as in Theorem 3.1. Then $X(\tilde{\pi}_{m_0^1}(g)) = X(\tilde{\pi}_{m_0^2}(g))$ for all $g \in \Gamma$ (denoted by $X_0(g)$). Moreover, $X(x, \rho_0(g)) = X_0(g)$ for all $g \in \Gamma$, m_0-almost every $x \in M$.*
(2) *There exists a neighborhood η of ρ_0 in $R(\Gamma, \mathrm{Diff}^1(M))$ such that for any $\rho \in \eta$, any ρ-invariant probability measure m, any $g \in \Gamma$ and m-almost every $x \in M$, $X(x, \rho(g)) = X_0(g)$.*

We have the following interesting corollary that asserts the rigidity of entropy for individual elements in the group.

THEOREM 3.4: *Let G and Γ be as before and ρ_0 be a C^r-action of G (or Γ) on a smooth compact manifold M with point Mather spectrum, and preserving some Lebesgue measure. Let ρ be a C^r action sufficiently C^1-close to ρ_0. Then:*

(1) *If $r \geq 1$, there exists a ρ-invariant absolutely continuous measure m;*
(2) *If $r > 1$ and m is an absolutely continuous ρ-invariant measure, then $h_m(\rho(g)) = h(\rho_0(g))$ for all $g \in G$ (or $g \in \Gamma$). Here $h(\rho_0(g))$ is both the topological entropy and the measure-theoretic entropy (with respect to the invariant Lebesgue measure) for $\rho_0(g)$.*

Proof: (1) Since G, Γ have property T (III.5.6, III.5.7 of [Mar]) and ρ is sufficiently C^1-close to ρ_0, ρ preserves an absolutely continuous measure m (Lemma 2.6 of [KLZ] for Γ; a similar argument works for G). We remark that the measure is not necessarily equivalent to the measure given by a smooth volume.

(2) Let m be any absolutely continuous ρ-invariant measure. Then Pesin's formula (proved by Pesin for volume preserving diffeomorphisms [Pe2]; extended to diffeomorphisms preserving absolutely continuous measure by others, see, for

example, [LS], [Mañ]) together with Theorems 3.2, 3.3 assert that for all $g \in G$ (or Γ), $h_m(\rho(g)) = \sum_{\chi_i>0} \chi_i$ (χ_i are from the multiset $X_0(g)$).

We show that $h(\rho_0(g)) := \sum_{\chi_i>0} \chi_i$ is both the topological entropy and the measure-theoretic entropy (with respect to the invariant Lebesgue measure) for $\rho_0(g)$. Indeed, by another application of Pesin's formula we obtain that $h(\rho_0(g))$ is the measure-theoretic entropy for $\rho_0(g)$ with respect to the invariant measure. By a variational principle (see, for example, Theorem 8.6 and Corollary 8.6.1 of [W]), the topological entropy of $\rho_0(g)$ is

$$\begin{aligned} h_{top}(\rho_0(g)) &= \sup\{h_\mu(\rho_0(g)) \colon \mu \text{ is } \rho_0(g)\text{-invariant probability measure}\} \\ &= \sup\{h_\mu(\rho_0(g)) \colon \mu \text{ is } \rho_0(g)\text{-invariant ergodic probability measure}\}. \end{aligned}$$

Therefore we have $h_{top}(\rho_0(g)) \geq h(\rho_0(g))$. For any $\rho_0(g)$-invariant ergodic measure μ, let $\chi_1^{(\mu)}, \ldots, \chi_n^{(\mu)}$ (necessarily constants) be the Lyapunov exponents of $\rho_0(g)$ with respect to μ. Then $h_\mu(\rho_0(g)) \leq \sum_{\chi_i^{(\mu)}>0} \chi_i^{(\mu)}$ by Ruelle's inequality [Ru2]. Since $\rho_0(g)$ has point Mather spectrum, the multiset $\{\chi_1^{(\mu)}, \ldots, \chi_n^{(\mu)}\}$ is independent of the choice of the $\rho_0(g)$-invariant ergodic probability measure μ (in fact it is equal to $X_0(g)$). Hence $h_\mu(\rho_0(g)) \leq \sum_{\chi_i>0} \chi_i = h(\rho_0(g))$. It follows that $h_{top}(\rho_0(g)) \leq h(\rho_0(g))$. Hence $h_{top}(\rho_0(g)) = h(\rho_0(g))$. ∎

References

[BP] M. I. Brin and Ya. B. Pesin, *Partially hyperbolic dynamical systems*, Mathematics of the USSR Izvestiya **8** (1974), 177–218.

[Fu] H. Furstenberg, *Rigidity and cocycles for ergodic actions of semisimple groups [after G.A. Margulis and R. Zimmer]*, Seminaire Bourbaki, No. 559, 1979/80.

[KLZ] A. Katok, J. Lewis and R. Zimmer, *Cocycle superrigidity and rigidity for lattice actions on tori*, Topology **35** (1996), 27–38.

[LS] F. Ledrappier and J.-M. Strelcyn, *A proof of the estimation from below in Pesin's entropy formula*, Ergodic Theory and Dynamical Systems **2** (1982), 203–219.

[Le] J. Lewis, *The algebraic hull of the derivative cocycle*, Preprint, 1993.

[Mañ] R. Mañé, *A proof of Pesin's formula*, Ergodic Theory and Dynamical Systems **1** (1981), 95–102.

[Mat] J. Mather, *Characterization of Anosov diffeomorphisms*, Indagationes Mathematicae **30** (1968), 479–483.

[Mar] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Springer-Verlag, New York, 1991.

[O] V. I. Oseledec, *A multiplicative ergodic theorem*, Transactions of the Moscow Mathematical Society **19** (1968), 197–231.

[Pe1] Ya. B. Pesin, *On the existence of invariant fiberings for a diffeomorphism of a smooth manifold*, Mathematics of the USSR-Sbornik **20** (1973), 213–222.

[Pe2] Ya. B. Pesin, *Characteristic Lyapunov exponents and smooth ergodic theory*, Russian Mathematical Surveys **32** (1977), 55–114.

[Ru1] D. Ruelle, *Ergodic theory of differentiable dynamical systems*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **50** (1979), 275–306.

[Ru2] D. Ruelle, *An inequality for the entropy of differentiable maps*, Boletim da Sociedade Brasileira de Matemática **9** (1978), 83–87.

[W] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, New York, 1981.

[Z1] R. J. Zimmer, *Orbit equivalence and rigidity of ergodic actions of Lie groups*, Ergodic Theory and Dynamical Systems **1** (1981), 237–253.

[Z2] R. J. Zimmer, *Volume preserving actions of lattices in semisimple groups on compact manifolds*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **59** (1984), 5–33.

[Z3] R. J. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhauser, Boston, 1984.

[Z4] R. J. Zimmer, *Actions of semisimple groups and discrete subgroups*, Proceedings of the International Congress of Mathematicians, Berkeley, 1986, pp. 1247–1258.

[Z5] R. J. Zimmer, *On the algebraic hull of an automorphism group of a principal bundle*, Commentarii Mathematici Helvetici **65** (1990), 375–387.